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# Motion of Bubbles in a Varying Pressure Field

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## I. Introduction

RECENTLY, there has been a growing interest in the properties of bubbly liquids. This interest ranges from the propagation of sound waves in liquids to the properties of bubbly liquid metals for various technological applications.[1-3,8] The dynamics of compressible bubbles in a liquid exhibit variations on two distinct time scales under various conditions. This behavior is caused by the natural volume oscillations of the bubbles, which are much faster than the motion of the liquid. As a result, the time step used in numerical calculations of bubbly flows is limited by the low period of the fast oscillations. This limitation can be removed by separating the flow variables into slow and fast components. By analytically finding the fast behavior and integrating over it, the time step can be significantly increased and is limited only by the much slower motion.

The separation of the flow variables into fast and slow components is carried out by the multiple-scale analysis.[4] The slow variations in the oscillation period and amplitude of the bubbles are determined by the requirement that no secular terms exist in any order of the equations of motion.

In Sec. II the behavior of a single bubble in a slowly varying pressure field is investigated. In Sec. III the equations of motion of bubbly flow are separated into slow variables and fast oscillations. The resulting equations are averaged over the fast time scale.

## II. Single-Bubble Dynamics

The equation describing the dynamics of a single bubble in an unbounded liquid is given by[5]

$$R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2=\frac{P_g-P_\infty}{\rho_1} \tag{1}$$

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where $R$ is the radius of the bubbles, $P_g$ its internal pressure, $\rho_1$ is the density of the liquid, and $P_\infty$ its pressure at infinity. It was assumed in Eq. (1) that the surface tension terms are small. Furthermore, in the case of a bubbly flow with a low gas content and small relative velocity between the bubbles and the liquid, Eq. (1) can still be used where $P_\infty$ is replaced by the liquid's pressure next to the bubble $P$. Whereas Eq. (1) describes the fast oscillations of the bubbles that are on time scale $t_b$, the pressure $P$ varies on a much slower time scale denoted by $t_1$. Thus Eq. (1) is given by

$$R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2=\frac{P_g-P(\tau)}{\rho_1} \tag{2}$$

where $\tau=\epsilon t$ and

$$\epsilon\approx t_b/t_1 \tag{3}$$

For a steady-state flow, $\epsilon$ can be estimated according to

$$\epsilon\approx\frac{u_b\,|\nabla P|}{P\omega_0} \tag{4}$$

where $u_b$ and $\omega_0$ are the velocity and natural frequency of the bubbles, respectively, and $P/|\nabla P|$ is an estimation of a characteristic length for a change in the liquid's pressure.

For the internal pressure $P_g$ we use the equation of state of an ideal gas. Thus

$$P_g/\rho_1=a/R^3 \tag{5}$$

where $a$ is a constant that depends on the temperature and the mass of the bubble. As was demonstrated by Plesset and Hsieh,[6] Eq. (5) holds for a wide range of frequencies.

Before turning to the multiple-scale analysis, we notice that since the bubble is expected to oscillate with a varying frequency, it is of benefit to define the following new variable:

$$\zeta=\int_0^t\omega_0(\epsilon\xi)\,d\xi=\frac{1}{\epsilon}\int_0^t\omega_0(\eta)\,d\eta \tag{6}$$

where the natural frequency is given by

$$\omega_0^2(\tau)=3a/R_0^5(\tau) \tag{7}$$

and $R_0(\tau)$ is obtained by setting the right-hand side of Eq. (2) to zero and using Eq. (5). The natural frequency at a constant pressure is obtained by linearizing Eq. (2) around a static state. After using the new independent variable defined by Eq. (6), Eq. (2) is transformed into the following equation:

$$R\frac{d^2R}{d\zeta^2}+\epsilon\frac{\omega_0'}{\omega_0^2}R\frac{dR}{d\zeta}+\frac{3}{2}\left(\frac{dR}{d\zeta}\right)^2=\frac{1}{\omega_0^2}\left(\frac{a}{R^3}-\frac{P}{\rho_1}\right) \tag{8}$$

where the prime denotes differentiation with respect to the argument.

We apply now the multiple-scale analysis. For this purpose we expand Eq. (8) in terms of the two time scales $\zeta$ and $\tau$ in the following way:

$$R(\zeta)=R_0(\tau)+\epsilon R_1(\zeta,\tau)+\epsilon^2R_2(\zeta,\tau)+\ldots \tag{9}$$

$$\frac{dR}{d\zeta}=\frac{\partial R}{\partial\zeta}+\frac{\epsilon}{\omega_0}\frac{\partial R}{\partial\tau} \tag{10}$$

$$\frac{d^2R}{d\zeta^2}=\frac{\partial^2R}{\partial\zeta^2}+2\frac{\epsilon}{\omega_0}\frac{\partial^2R}{\partial\zeta\partial\tau}+\frac{\epsilon^2}{\omega_0^2}\frac{\partial^2R}{\partial\tau^2} \tag{11}$$

Equations (9-11) are inserted into Eq. (8), and the resulting equation is solved order by order in $\epsilon$. The zero order is automatically satisfied by the definition of $R_0(\tau)$. The first-order equation is given by

$$\frac{\partial^2 R_1}{\partial \zeta^2}+R_1=0 \tag{12}$$

The solution of Eq. (12) is given by

$$R_1(\zeta,\tau)=A(\tau)\cos\zeta+B(\tau)\sin\zeta \tag{13}$$

The coefficients $A$ and $B$, which vary slowly in time, are determined by the requirement that the equation for $R_2$ does not include terms which might give rise to secular behavior. Thus the second-order equation is given by

$$\frac{\partial^2 R_2}{\partial \zeta^2}+R_2=-\left[\frac{2R_0}{\omega_0}\frac{\partial^2 R_1}{\partial\zeta\partial\tau}+\frac{\omega_0'}{\omega_0}R_0\frac{\partial R_1}{\partial \zeta}+\frac{3}{2}R_0'\frac{\partial R_1}{\partial\zeta}\right]+\text{NST} \tag{14}$$

The first three terms on the right-hand side of Eq. (14) contain all the terms that might give rise to a secular behavior of $R_2$ while the rest (nonsecular terms) contain $\cos 2\zeta$ and $\sin 2\zeta$. Thus the slowly varying coefficients are determined now by the requirement that the three first terms vanish:

$$2\frac{A'}{A}=-\frac{\omega_0'}{\omega_0}-3\frac{R_0'}{R_0} \tag{15}$$

and the same equation for $B$. The solution of Eq. (15) is given by

$$A(\tau)=-A_0/(\omega_0^{1/2}R_0^{3/2}) \tag{16}$$

Using Eq. (6), we finally write

$$R_1(\zeta,\tau)=\frac{1}{[R_0(\tau)]^{1/4}}\left\{A_0\cos\left[\frac{1}{\epsilon}\int_0^\tau\omega_0(x)\mathrm{d}x\right]+B_0\sin\left[\frac{1}{\epsilon}\int_0^\tau\omega_0(x)\mathrm{d}x\right]\right\} \tag{17}$$

where $A_0$ and $B_0$ are constants determined by the initial conditions.

An important result is immediately obvious, namely that if an oscillating bubble is moving into an area of increasing pressure, its equilibrium radius decreases, and hence the amplitude of the oscillations is enhanced. On the other hand, the oscillations of a bubble moving in the opposite direction of the pressure gradient are stable.

## III. Averaged Bubbly Flow

The method described in the previous section can be used to eliminate the fast time scale from the equations that govern the motion of bubbles in a liquid. In order to show that, the following assumption are made: 1) the liquid is incompressible, 2) the bubbles move in the liquid with a constant velocity, 3) the number of bubbles per unit volume $n_0$ is a constant, 4) the relative velocity between the bubbles and the liquid is small, hence the bubbles are assumed to be spherical, and 5) the average distance between the bubbles is small, hence Eq. (1) can be used for describing the bubbles' radius. It is noted that not all the assumptions previously listed are necessary for the following discussion. Assumptions 1-3 are made in order to make the algebra more transparent and can be removed for a more general case. Under these assumptions, the equations that govern the motion of the bubbles in the liquid are

$$\frac{\partial}{\partial t}(1-\alpha)+\frac{\partial}{\partial x}[(1-\alpha)u]=0 \tag{18}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{\rho_1}\frac{\partial P}{\partial x} \tag{19}$$

$$R\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2=\frac{a}{R^3}-\frac{P}{\rho_1} \tag{20}$$

where $\alpha$ is the void fraction and is given by $4\pi n_0 R^3/3$. A model similar to Eqs. (18-20) was introduced by Wijngaarden[7] and discussed by Caflisch et al.[8]

As in Sec. II, a new independent variable $\zeta$ is defined by Eq. (6). In addition, a new space variable is defined by

$$\xi^*=\int_0^{x/u_b}\omega_0(\epsilon x)\mathrm{d}x \tag{21}$$

It is more convenient, however, to transform to the frame of reference moving with the bubbles by

$$\xi=\xi^*-\zeta \qquad v=(u-u_b)/u_b \tag{22}$$

In terms of the new variables, Eqs. (18-20) become

$$\frac{\partial\alpha}{\partial\zeta}+v\frac{\partial\alpha}{\partial\xi}-(1-\alpha)\frac{\partial v}{\partial\xi}=0 \tag{23}$$

$$\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial\xi}=-\frac{1}{\rho_1}\frac{\partial P}{\partial\xi} \tag{24}$$

and Eq. (8). Similar to the temporal behavior of a single bubble, the spatial behavior is split into fast and slow variations. Thus a slow coordinate is defined by

$$\sigma=\epsilon\xi \tag{25}$$

The following solutions are assumed for the dependent variables:

$$\psi=\psi_0(\sigma,\tau,\epsilon)+\sum_{n=1}^{\infty}\epsilon^n\psi(\sigma,\tau,\xi,\epsilon)\exp(in\zeta) \tag{26}$$

The averaging process works as follows: Eq. (26) is inserted into Eqs. (23), (24), and (8). The resulting equations are then multiplied by $\exp(in\zeta)$ and integrated over a whole period in $\zeta$. Each of the resulting equations is then solved order by order in $\epsilon$. The lowest-order equations are

$$\frac{\partial\alpha_0}{\partial\tau}+v_0\frac{\partial\alpha_0}{\partial\tau}-(1-\alpha_0)\frac{\partial v_0}{\partial\sigma}=0 \tag{27}$$

$$\frac{\partial v_0}{\partial\tau}+v_0\frac{\partial v_0}{\partial\sigma}=-\frac{1}{\rho_1}\frac{\partial P_0}{\partial\sigma} \tag{28}$$

where $R_0$ and $\alpha_0$ are given by

$$R_0^3(\tau,\sigma,\epsilon)=a\rho_1/[P_0(\tau,\sigma,\epsilon)] \tag{29}$$

$$\alpha_0=(4\pi/3)n_0(R_0^3+3\epsilon^2R_0R_1^2)+\mathcal{O}(\epsilon^3) \tag{30}$$

It is easy to show that the first-order solution that appears in Eq. (30) is given by

$$R_1 = R_1(\tau,\sigma,\epsilon)\exp(i\upsilon_0\xi) \quad (31)$$

and $R_1$ is calculated in a similar way as outlined in Sec. II.

From Eqs. (27-31), we notice two important results. First, the equations of motion that govern the bubbly flow are written in terms of $\tau$ and $\sigma$ alone, which represent the slow temporal and spatial variations respectively. Thus, solving Eqs. (27) and (28) numerically, a larger time step can be used which is $1/\epsilon$ bigger than the time step needed for solving the original set of equations (18-20). Second, the fast oscillations affect the bulk motion of the two phases at least in order $\epsilon^2$.

## IV. Conclusions

The existence of two distinct time scales in bubbly flows was used in order to integrate over the fast variations that occur due to the natural volume oscillations. First, the oscillations of a single bubble in a slowly varying pressure field were investigated. Using the multiple-scale technique it was shown that these oscillations are enhanced if the liquid's pressure is growing. Then, a procedure was introduced that averages the equations of bubbly flows over the fast oscillations.

As a result, a set of reduced differential equations is obtained which depend only on the slowly varying independent variables. This procedure enables the use of large time steps in numerical computations.

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# Optimization of Equivalent Periodic Truss Structures

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## Introduction

SIMPLE beam models of complex periodic space structures can be effectively used to optimize the dimensions of the structural members. This could result in enormous cost savings as well as provide the analyst with valuable insights into the behavior of complex periodic space structures. However, care must be exercised when computing member loads from these simple beam models. Specifically, for a long cantilever periodic space structure, the fundamental frequency and tip displacement can be accurately determined using a Bernoulli-Euler beam. However, the member loads can be accurately calculated only by using a Timoshenko beam.

## Behavior of Truss Structure

The cantilever truss shown in Fig. 1 is taken to illustrate the difference obtained by using a Bernoulli-Euler beam and a Timoshenko beam. The member properties are as follows:

$E = 71.7\times10^9$ N/m$^2$ $\quad$ $L = 75$ m

$\rho = 2768$ kg/m$^3$ $\quad$ $A_c = 80\times10^{-6}$ m$^2$

$L_c = 7.5$ m $\quad$ $A_g = 60\times10^{-6}$ m$^2$

$L_g = 5.0$ m $\quad$ $A_d = 40\times10^{-6}$ m$^2$

where $A_c$, $A_g$, and $A_d$ are the cross-sectional areas of the vertical, horizontal, and diagonal members, respectively. These dimensions and properties were chosen so that the truss could be considered a Bernoulli-Euler beam. One load condition, as shown in Fig. 1, is imposed: $P = 200$ N.

Using the approach developed by Sun et al.,[1] the equivalent beam properties can be obtained as follows:

$$AE = 2A_dE(\beta^3+\gamma) \quad (1)$$

$$EI = 0.5EA_cL_g^2 \quad (2)$$

$$KAG = 2A_dE\alpha^2(1-\alpha^2)^{1/2} \quad (3)$$

where

$$\alpha = L_g/L_d, \qquad \beta = L_c/L_d, \qquad \gamma = A_c/A_d$$

The design problem to be solved here can be stated as follows: Find the dimension of the members such that the mass of the structure

$$m = \{2n[A_cL_c + A_dL_d] + (n+1)[A_gL_g]\}\rho \quad (4)$$

where $n$, the number of periodic structures, is minimized subject to the following constraints:

1) Frequency Constraint

$$(\omega/\omega_0) - 1 \geq 0 \quad (5)$$

where $\omega$ is the fundamental frequency of a Bernoulli-Euler beam and is given by

$$\omega = [1.875/L]^2\sqrt{EIL/m_r} \quad (6)$$

and $m_r = m - A_gL_g\rho$ (since base girder fixed mass is not participating). Also, $\omega_0$ is the minimum allowable fundamental frequency, which for this problem is 5 rad/s.

2) Tip Displacement Constraint

$$1-(\Delta_x/\Delta_{max}) \geq 0 \quad (7)$$

where $\Delta_x$ is the tip displacement given by

$$\Delta_x = PL^3/3EI \qquad \text{(Bernoulli-Euler)}$$

$$\Delta_x = (PL^3/3EI) + (PL/KAG) \qquad \text{(Timoshenko)} \quad (8)$$

Also, $\Delta_{max}$ is the maximum allowable tip displacement, which for this problem is 0.4 m.

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